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# Recursively Arbitrarily Vertex-Decomposable Graphs\*

Olivier Baudon<sup>a</sup>, Frédéric Gilbert<sup>a</sup>, Mariusz Woźniak<sup>b†</sup>

<sup>a</sup>Univ. Bordeaux, LaBRI, UMR 5800, F-33400 Talence, France

CNRS, LaBRI, UMR 5800, F-33400 Talence, France

<sup>b</sup>Faculty of Applied Mathematics, AGH University of Science and Technology  
Al. Mickiewicza 30, 30-059 Krakow, Poland

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## Abstract

A graph  $G = (V, E)$  is arbitrarily vertex decomposable if for any sequence  $\tau$  of positive integers adding up to  $|V|$ , there is a sequence of vertex-disjoint subsets of  $V$  whose orders are given by  $\tau$ , and which induce connected graphs. The main aim of this paper is to study the recursive version of this problem. We present a solution for trees, suns, and partially for a class of 2-connected graphs called balloons.

## 1 Introduction

We deal with finite, simple and undirected graphs. Let  $G = (V, E)$  be a graph of order  $n$ . A sequence  $\tau = (n_1, \dots, n_k)$  of positive integers is called *admissible for  $G$*  if it sums up to  $n$ . If  $\tau = (n_1, \dots, n_k)$  is an admissible sequence for  $G$  and there exists a partition  $(V_1, \dots, V_k)$  of the vertex set  $V$  such that  $V_i$  induces a connected subgraph of order  $n_i$  for each  $i \in [1, k]$ , then  $\tau$  is called *realizable in  $G$* , and the sequence  $(V_1, \dots, V_k)$  is said to be *a realization of  $\tau$  in  $G$* . A graph  $G$  is *arbitrarily vertex decomposable* (AVD, for short) if for each admissible sequence  $\tau$  for  $G$  there exists a realization of  $\tau$  in  $G$ . Clearly, every AVD graph has to be connected.

The notion of AVD graphs was first introduced by Barth et al. in [1] motivated by the following problem in computer science. Consider a network connecting different computing resources; such a network is modeled by a graph. Suppose there are  $k$  different users, where the  $i$ -th one needs  $n_i$  resources in our graph. The subgraph induced by the set of resources attributed to each user should be connected and each resource should be attributed to one user. So we are seeking a realization of the sequence  $\tau = (n_1, \dots, n_k)$  in this graph. Thus, such a network should be an AVD graph.

This problem can also be considered as a natural analogy of the similar notion in which vertices are replaced by edges. Some references concerning arbitrarily edge decomposable graphs can be found in [8].

The problem of deciding whether a given graph is arbitrarily vertex decomposable has been considered in several papers. The investigation of AVD trees is motivated by the fact that a connected graph is AVD if its spanning tree is AVD. It turned out, however, that the structure of AVD trees is not obvious in general.

On the other hand, it is clear that each path, and therefore each traceable graph, is AVD. So, the problems concerning AVD graphs can be considered as a generalization of hamiltonian problems.

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‡email addresses: {olivier.baudon, frederic.gilbert}@labri.fr, mwozniak@agh.edu.pl

Some results in this direction were obtained by use of some characterizations of AVD graphs in the family of graphs with a large dominating cycle, called suns. This spurs on and justifies investigating AVD graphs in this family.

## 2 Terminology and preliminary results

In this paper, we deal only with simple graphs, that means graphs without loops or multiple edges. We denote by  $n$  the number of vertices, also called *order* of the graph and by  $m$  the number of edges. If  $G = (V, E)$  and  $A \subseteq V$ ,  $G[A]$  will denote the subgraph of  $G$  induced by  $A$ . For more definitions on graphs, please refer to [6].

### 2.1 Arbitrarily Vertex-Decomposable Graphs

Let  $n, \tau_1, \dots, \tau_k$  be positive integers such that  $\tau_1 + \dots + \tau_k = n$ .  $\tau = (\tau_1, \dots, \tau_k)$  is called a *decomposition* of  $n$ .

Let  $G = (V, E)$  be a graph of order  $n$ , and  $\tau = (\tau_1, \dots, \tau_k)$  a decomposition of  $n$ .  $G$  is  $\tau$ -*Vertex-Decomposable* iff there exists a partition of  $V : V_1, \dots, V_k$  such that for each  $i, 1 \leq i \leq k$ ,

- $|V_i| = \tau_i$ ,
- $G[V_i]$  is connected.

A graph  $G = (V, E)$  of order  $n$  is *Arbitrarily Vertex-Decomposable* (for short *AVD*) iff for each decomposition  $\tau$  of  $n$ ,  $G$  is  $\tau$ -Vertex-Decomposable.

### 2.2 Recursively Arbitrarily Vertex-Decomposable Graphs

We present here a new definition of graph decomposition into connected components. This definition introduces the main topic of this paper. The goal is to characterize graphs for which it is possible to perform arbitrarily decomposition recursively on each subgraph.

**Definition 1** A graph  $G = (V, E)$  of order  $n$  is *Recursively Arbitrarily Vertex-Decomposable* (for short *R-AVD*) iff

- $G = K_1$   
or
- $G$  is connected and for each decomposition  $\tau = (\tau_1, \dots, \tau_k)$  of  $n$ ,  $k \geq 2$ , there exists a partition of  $V : V_1, \dots, V_k$  such that for all  $i, 1 \leq i \leq k$ ,
  - $|V_i| = \tau_i$ ,
  - $G[V_i]$  is *R-AVD*.

A stronger version of the recursive decomposition of graphs into connected components will also be studied in the last section of this paper, based on the following definition:

**Definition 2** A graph  $G = (V, E)$  of order  $n$  is *Strongly Recursively Arbitrarily Vertex-Decomposable* (for short *SR-AVD*) iff

- $G$  is *AVD*,
- for each sequence  $\tau$  of length  $k \geq 2$  admissible for  $G$  and for each realization  $V_1, \dots, V_k$  of  $\tau$  the graph  $G[V_i]$ ,  $i = 1, \dots, k$ , is *SR-AVD*.

## 2.3 Families of graphs

The following definitions presents some families of graphs and their notations, used in the further sections.

### Definition 3

- Let  $a$  be a positive integer.  $P_a$  denotes the path of order  $a$ ,  $C_a$  the cycle of order  $a$  (cf. Figures 1a and 1b) .
- A  $k$ -pode  $T_k(t_1, \dots, t_k)$  is a tree of order  $1 + \sum_{i=1}^k t_i$  composed by  $k$  paths of respective orders  $t_1, \dots, t_k$ , connected to a unique node, called the root of the  $k$ -pode (cf. Figure 1c).
- Let  $a$  and  $b$  be two positive integers. A caterpillar  $\text{Cat}(a, b)$  is a tree of order  $a + b$ , composed by three paths of order  $a$ ,  $b$  and 2, sharing exactly one node, called the root of the caterpillar.  $\text{Cat}(a, b)$  is isomorphic to  $T_3(a-1, b-1, 1)$  (cf. Figure 1d).
- A sun with  $r$  rays is a graph of order  $n \geq 2r$  with  $r$  hanging vertices  $u_1, \dots, u_r$  whose deletion yields a cycle  $C_{n-r}$ , and each vertex  $v_i$  adjacent to  $u_i$  is of degree three. If the sequence of vertices  $v_i$  is situated on the cycle  $C_{n-r}$  in such a way that there are exactly  $a_i \geq 0$  vertices, each of degree two, between  $v_i$  and  $v_{i+1}$ ,  $i = 1, \dots, r$  (the indices taken modulo  $r$ ), then this sun is denoted by  $\text{Sun}(a_1, \dots, a_r)$ , and is unique up to isomorphism (cf. Figure 1e).  
Note that the order of  $\text{Sun}(a_1, \dots, a_r)$  equals  $n = 2r + a_1 + \dots + a_r$ .
- Let  $b_1, \dots, b_k$  be positive integers. A  $k$ -balloon  $B(b_1, \dots, b_k)$  is a graph of order  $2 + \sum_{i=1}^k b_i$  composed by two vertices (called roots) linked by  $k$  paths (called branches) of widths (the number of internal vertices)  $b_1, \dots, b_k$  (cf. Figure 1f).

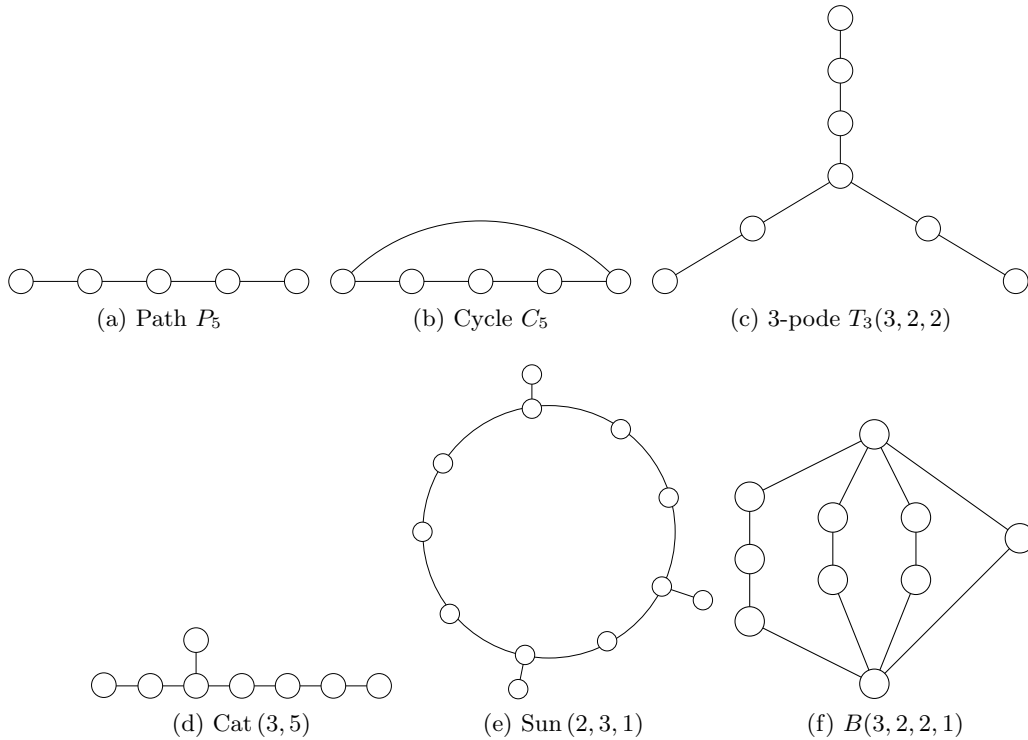


Figure 1: Examples of Graphs

**Definition 4**

A graph of order  $n$  even (resp. odd) has a perfect matching (resp. quasi-perfect matching) iff it contains a set of  $\lfloor \frac{n}{2} \rfloor$  disjoint edges.

A graph is traceable iff it contains an hamiltonian path.

## 2.4 Preliminary results

**Remark 5** A graph  $G$  having an AVD spanning subgraph is AVD. But there are AVD-graphs without any AVD-spanning tree.

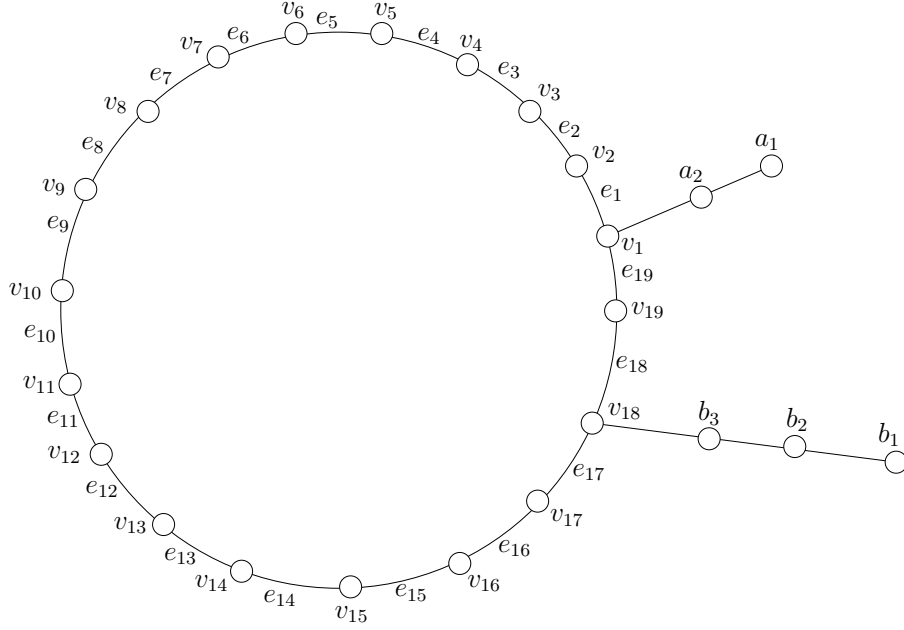


Figure 2: An AVD graph without any AVD spanning tree

**Proof.** The first part of Remark 5 is trivial. We now prove that the graph  $G$  of Figure 2<sup>1</sup> is AVD, but has no AVD spanning tree.

To prove that the graph is AVD, we show that for any value of  $\lambda \leq n-1$ , we may find a connected induced subgraph  $G[V_\lambda]$  of  $G$  of order  $\lambda$  such that the remaining graph  $G[V \setminus V_\lambda]$  is traceable. The values of  $V_\lambda$  are given in Table 1.

$\lambda$	$V_\lambda$
1	$\{v_{19}\}$
2	$\{a_1, a_2\}$
3	$\{b_1, b_2, b_3\}$
4	$\{a_1, a_2, v_1, v_{19}\}$
$\geq 5$	$\{a_1, a_2, v_{19}, v_1, v_2, \dots, v_{\lambda-3}\}$

Table 1: Values of  $V_\lambda$

<sup>1</sup>This counter-example has been proposed by Hervé Fournier. Another counter-example may be found in [12].

The graph  $G$  is of order  $n = 24$  and any of its spanning trees is of the form  $G - e_j$ ,  $j \in \{1, \dots, 19\}$ . If  $G - e_j$  is AVD, then the sequence  $(i, \dots, i)$  is realizable in  $G - e_j$ ,  $i = 2, 3, 4, 6$ . For  $i \in \{2, 3, 4, 6\}$  let  $R_i$  be the set of all  $j \in \{1, \dots, 19\}$  such that  $(i, \dots, i)$  is realizable in  $G - e_j$ . Then  $R_2 = \{1, 2, 3, 5, 7, 9, 11, 13, 15, 17, 18, 19\}$ ,  $R_3 = \{1, 4, 7, 10, 13, 16, 19\}$ ,  $R_4 = \{1, 5, 9, 13, 17\}$  and  $R_6 = \{3, 4, 9, 10, 15, 16, 18, 19\}$ . Since  $R_2 \cap R_3 \cap R_4 \cap R_6 = \emptyset$ , the graph  $G$  has no AVD spanning tree.  $\square$

**Remark 6** A graph  $G = (V, E)$  of order  $n$  is  $R$ -AVD iff for each integer  $1 \leq \lambda \leq \lfloor \frac{n}{2} \rfloor$ , it exists a subset  $V_\lambda$  of  $V$  such that

- $|V_\lambda| = \lambda$ ,
- $G[V_\lambda]$  is  $R$ -AVD,
- $G[V \setminus V_\lambda]$  is  $R$ -AVD.

The notion of on-line arbitrarily vertex decomposable graph has been introduced by Hornák et al. in [7]. This version of the problem is even more natural when applied to the problem in computer networks mentioned in Section 1.

Let  $G = (V, E)$  be a graph. Imagine now the following decomposition procedure consisting of  $k$  stages, where  $k$  is a random variable attaining (integer) values from interval  $[1, n]$ . In the  $i^{th}$  stage, where  $i \in [1, k]$ , a positive integer  $\tau_i$  arrives and we have to choose a subset  $V_i$  of  $V$  of order  $\tau_i$  that is disjoint from all subsets of  $V$  chosen in previous stages (without a possibility of changing the choice in the future).

More precisely, for every partial sequence  $(\tau_1, \dots, \tau_i)$  whose sum is less than  $n$ , there is a sequence  $(V_1, \dots, V_i)$  of disjoint subsets of  $V$  such that for  $1 \leq j \leq i$ ,  $|V_j| = \tau_j$ , with the following property: for all sequences  $(\tau'_1, \dots, \tau'_k)$  with  $k \geq i$  and summing to  $n$ , such that  $\tau'_r = \tau_r$  for  $1 \leq r \leq i$ , there is a decomposition of  $V$  into disjoint subsets  $V'_1, \dots, V'_k$  with  $|V'_j| = \tau'_j$  and  $G[V'_j]$  connected, for all  $j$ , and  $V'_j = V_j$  for  $1 \leq j \leq i$ .

**Definition 7** [7] If the decomposition procedure can be accomplished for any (random) sequence of positive integers  $(\tau_1, \dots, \tau_k)$  adding up to  $n$ , the graph  $G$  is said to be On-Line AVD, (for short OL-AVD).

**Lemma 8** [7] A graph connected  $G = (V, E)$  of order  $n$  is OL-AVD iff for each integer  $1 \leq \lambda \leq n-1$ , there exists a subset  $V_\lambda$  of  $V$  such that

- $|V_\lambda| = \lambda$ ,
- $G[V_\lambda]$  is connected,
- $G[V \setminus V_\lambda]$  is OL-AVD.

**Remark 9** A straightforward consequence of Lemma 8 and Remark 6 is that every  $R$ -AVD graph is OL-AVD.

The opposite is not true. For example, the caterpillar  $\text{Cat}(8, 11)$  is OL-AVD [7], but not  $R$ -AVD (cf. Section 3.1).

We denote by :

- $PF(n)$  the set of graphs of order  $n$  with a perfect matching or a quasi-perfect matching ;
- $AVD(n)$  the set of AVD graphs of order  $n$  ;
- $OL-AVD(n)$  the set of OL-AVD graphs of order  $n$  ;
- $R-AVD(n)$  the set of  $R$ -AVD graphs of order  $n$  ;
- $\text{Traceable}(n)$  the set of traceable graphs of order  $n$  ;
- $SR-AVD(n)$  the set of SR-AVD graphs of order  $n$  ;

**Theorem 10**  $PF(n) \supsetneq AVD(n) \supsetneq OL-AVD(n) \supsetneq R-AVD(n) \supsetneq Traceable(n) \supsetneq SR-AVD(n)$

**Proof.**

- $PF(n) \supsetneq AVD(n)$

If  $G$  is AVD, then it is  $(2, \dots, 2)$  or  $(2, \dots, 2, 1)$ -decomposable, following the parity of its order.

The caterpillar  $Cat(3, 3)$  has a perfect matching, but is not  $(3, 3)$ -decomposable.

- $AVD(n) \supsetneq OL-AVD(n)$

The inclusion is trivial.

The 3-pode  $T_3(2, 3, 5)$  is AVD ([1]), but not OL-AVD ([7]).

- $OL-AVD(n) \supsetneq R-AVD(n)$

cf. Remark 9.

- $R-AVD(n) \supsetneq Traceable(n)$

The inclusion is trivial, using Remark 6.

The caterpillar  $Cat(2, 3)$  is R-AVD (cf. Section 3.1) and not traceable.

- $Traceable(n) \supsetneq SR-AVD(n)$

The proof of the inclusion is given in Section 4, Theorems 26 and 27.

The graph from Figure 3 is traceable, but not SR-AVD. Consider the  $(4, 1)$  decomposition, where the set of size 4 is  $S = \{v_1, v_2, v_3, v_5\}$ . The induced subgraph  $G[S]$  is isomorphic to  $K_{1,3}$  and then is not  $(2, 2)$ -decomposable.

□

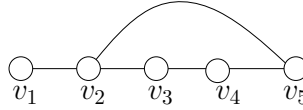


Figure 3: A traceable graph, which is not SR-AVD

## 2.5 Known results on AVD and OL-AVD graphs

**Theorem 11** [2] *There is no AVD  $k$ -pode for  $k \geq 5$ . All AVD 4-podes have a branch of length 1.*

A direct consequence of Theorem 11 is that the maximum degree of an AVD tree is at most 4. Nevertheless, this problem remains difficult on the class of trees, as shown by the next theorem.

**Definition 12** *Decision problem TP3 :*

*INSTANCE: A tree  $T$  with maximum degree 3 and order  $n$  and  $\tau$  a decomposition of  $n$ .*

*QUESTION: Is  $T$   $\tau$ -decomposable ?*

**Theorem 13** [2] *TP3 is NP-Complete.*

The two next results give a complete characterization of OL-AVD trees and suns.

**Theorem 14** [7] *A tree  $T$  is OL-AVD if and only if either  $T$  is a path or  $T$  is a caterpillar  $Cat(a, b)$  with  $a$  and  $b$  given in Table 2 or  $T$  is the 3-pode  $T_3(2, 4, 6)$ .*

$a$	$b$	$a$	$b$
2, 4	$\equiv 1 \pmod{2}$	7	8, 9, 11, 13, 15
3	$\equiv 1, 2 \pmod{3}$	8	11, 19
5	6, 7, 9, 11, 14, 19	9, 10	11
6	$\equiv 1, 5 \pmod{6}$	11	12

Table 2: Values  $a, b$  ( $b \geq a$ ), such that  $\text{Cat}(a, b)$  is OL-AVD

**Theorem 15** [10]

*A sun with one ray is always OL-AVD.*

*A sun with two rays  $\text{Sun}(a, b)$  is OL-AVD iff  $a$  and  $b$  take values given in Table 3a.*

*A sun with three rays  $\text{Sun}(a, b, c)$  is OL-AVD iff  $a, b$  and  $c$  take values given in Table 3b.*

*A sun with four rays is OL-AVD iff it is isomorphic to  $\text{Sun}(0, 0, 1, d)$ , where  $d \equiv 2, 4 \pmod{6}$ .*

*A sun with five or more rays is never OL-AVD.*

$a$	$b$	$a$	$b$	$c$
0	arbitrary	0	0	$\equiv 1, 2 \pmod{3}$
1, 3	$\equiv 0 \pmod{2}$		1	$\equiv 0 \pmod{2}$
2	$\not\equiv 3 \pmod{6}, 3, 9, 21$		2	$\equiv 2, 4 \pmod{6}, 3, 6, 7, 11, 18, 19$
4	$\equiv 2, 4 \pmod{6}, [4, 19] \setminus \{15\}$		3	$\equiv 2, 4 \pmod{6}$
5	$\equiv 2, 4 \pmod{6}, 6, 18$		4	4, 5, 6, 8, 10, 11, 12, 14, 16
6	6, 7, 8, 10, 11, 12, 14, 16		5	6, 8, 16
7	8, 10, 12, 14, 16		6, 7	8, 10
8	8, 9, 10, 11, 12		8	8, 9
9	10, 12	1	2	$\equiv 2, 4 \pmod{6}, 6, 18$
		2	3	4, 8, 16

(a) Values  $a, b$  ( $b \geq a$ ), such that  $\text{Sun}(a, b)$  is OL-AVD

(b) Values  $a, b, c$  ( $c \geq b \geq a$ ), such that  $\text{Sun}(a, b, c)$  is OL-AVD

Table 3: Values for OL-AVD Suns

### 3 Recursively AVD graphs

In this section, we give a complete characterization of R-AVD trees and suns and a tight upper bound of the maximum degree of R-AVD balloons.

#### 3.1 Trees

**Theorem 16** *A tree  $T$  is R-AVD if and only if either  $T$  is a path or  $T$  is a caterpillar  $\text{Cat}(a, b)$  with  $a$  and  $b$  given in Table 4 or  $T$  is the 3-pode  $T_3(2, 4, 6)$ .*

**Proof.** Only trees which are OL-AVD may be R-AVD (cf. Remark 9). The case of paths is trivial. Thus, we have only to check whether or not caterpillars from Table 2 and  $T_3(2, 4, 6)$  are R-AVD. For that, we use Remark 6. That means that for each  $\lambda \leq \frac{n}{2}$ , where  $n$  denotes the order of



$a$	$b$
2, 4	$\equiv 1 \pmod{2}$
3	$\equiv 1, 2 \pmod{3}$
5	6, 7, 9, 11, 14, 19
6	7
7	8, 9, 11, 13, 15

Table 4: Values  $a, b$  ( $b \geq a$ ), such that  $\text{Cat}(a, b)$  is R-AVD

the graph, we have to find a subset of vertices  $V_\lambda$  of size  $\lambda$  such that both  $G[V_\lambda]$  and  $G[V \setminus V_\lambda]$  are R-AVD.

Labellings are those from Figure 4.

For  $\text{Cat}(a, b)$  and  $\lambda = 1$ ,  $V_\lambda = \{x\}$ . The remaining graph is a path.

For  $\text{Cat}(a, b)$  and  $2 \leq \lambda \leq \frac{a+b}{2}$ , we have only two possibilities: either  $a_{a-1}$  or  $b_{b-1}$  belongs to  $V_\lambda$ . If  $\text{Cat}(a, b)$  is R-AVD, the two graphs  $G[V_\lambda]$  and  $G[V \setminus V_\lambda]$  are either a path or a smaller R-AVD caterpillar. Table 5 gives the solution when it exists. For values  $a$  and  $b$  such that  $\text{Cat}(a, b)$  is OL-AVD, but not R-AVD, Table 6 gives a value  $\lambda$  for which it is not possible to find a set  $V_\lambda$  such that both  $G[V_\lambda]$  and  $G[V \setminus V_\lambda]$  are R-AVD.

For  $T_3(2, 4, 6)$ , values of  $V_\lambda$  are given in Table 7, using labelling from Figure 4.  $\square$

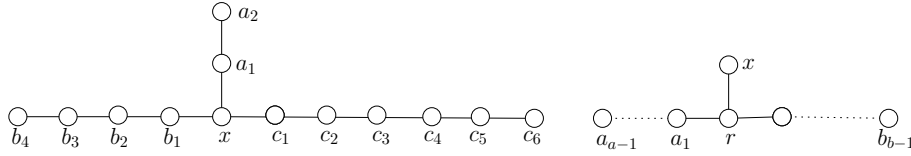


Figure 4: 3-pode  $T_3(2, 4, 6)$  and  $\text{Cat}(a, b)$

## 3.2 Suns

### Theorem 17

*A sun with one ray is always R-AVD.*

*A sun with two rays  $\text{Sun}(a, b)$  is R-AVD if and only if  $a$  and  $b$  take values given in Table 8a.*

*A sun with three rays  $\text{Sun}(a, b, c)$  is R-AVD if and only if  $a, b$  and  $c$  take values given in Table 8b.*

*A sun with four rays is R-AVD if and only if it is isomorphic to  $\text{Sun}(0, 0, 1, 2)$  or to  $\text{Sun}(0, 0, 1, 4)$ .*

*A sun with five or more rays is never R-AVD.*

The proof of this theorem uses arguments that are similar to those used for theorem 16. Because it is quite long, it has been given in a separate paper [4].

## 3.3 Balloons

### 3.3.1 Maximum degree of RAVD Balloons

**Remark 18**  $B(b_1, \dots, b_k)$  is AVD (resp. R-AVD) iff  $B(b_1, \dots, b_k, 0)$  is AVD (resp. R-AVD). Thus, we will always consider non-zero values for  $b_i, 1 \leq i \leq k$ .

$a$	$b$	$\lambda$	$\in V_\lambda$
any	any	1	$x$
2	$\equiv 1 \pmod{2}$	$\equiv 1 \pmod{2}$	$a_1$
		$\equiv 0 \pmod{2}$	$b_{b-1}$
3	$\equiv 1, 2 \pmod{3}$	$\equiv 1, 2 \pmod{3}$	$a_2$
		$\equiv 0 \pmod{3}$	$b_{b-1}$
4	$\equiv 1 \pmod{2}$	$\equiv 1 \pmod{2}$	$a_3$
		$\equiv 0 \pmod{2}$	$b_{b-1}$
5	6	4	$a_4$
		2, 3, 5	$b_5$
	7	2, 3, 4	$a_4$
		5, 6	$b_6$
	9	4	$a_4$
		2, 3, 5, 6, 7	$b_8$
	11	3, 4, 6	$a_4$
		2, 5, 7, 8	$b_{10}$

$a$	$b$	$\lambda$	$\in V_\lambda$	
5	14	2, 4, 6, 9	$a_4$	
		3, 5, 7, 8	$b_{13}$	
	19	2, 3, 4, 6, 7, 9, 11, 12	$a_4$	
5, 8, 10		$b_{18}$		
6	7	2, 3, 4, 5	$a_5$	
		6	$b_6$	
7	8	6	$a_6$	
		2, 3, 4, 5, 7	$b_7$	
	9	2, 5, 6	$a_6$	
		3, 4, 7, 8	$b_8$	
		2, 3, 4, 5, 6, 8	$a_6$	
	11	7, 9	$b_{10}$	
		13	3, 5, 6, 8	$a_6$
			2, 4, 7, 9, 10	$b_7$
	15	3, 5, 6, 8	$a_6$	
		2, 4, 7, 9 10	$b_{14}$	

Table 5: Leaf of  $\text{Cat}(a, b)$  in  $V_\lambda$  following values of  $a$ ,  $b$  and  $\lambda$

$a$	$b$	$\lambda$
6	$\equiv 1 \pmod{6}$ and $\neq 7$	9
	$\equiv 5 \pmod{6}$	8
8	11	2
	19	8

$a$	$b$	$\lambda$
9	11	3
10	11	4
11	12	2

Table 6: Values  $a$  and  $b$  with  $b \geq a$ , such that  $\text{Cat}(a, b)$  is OL-AVD, but not R-AVD

$\lambda$	$V_\lambda$	$G[V_\lambda]$	$G[V \setminus V_\lambda]$
1	$\{a_2\}$	$P_1$	$\text{Cat}(5, 7)$
2	$\{a_1, a_2\}$	$P_2$	$P_{11}$
3	$\{b_2, b_3, b_4\}$	$P_3$	$\text{Cat}(3, 7)$

$\lambda$	$V_\lambda$	$G[V_\lambda]$	$G[V \setminus V_\lambda]$
4	$\{b_1, b_2, b_3, b_4\}$	$P_4$	$P_9$
5	$\{c_2, c_3, c_4, c_5, c_6\}$	$P_5$	$\text{Cat}(3, 5)$
6	$\{c_1, c_2, c_3, c_4, c_5, c_6\}$	$P_6$	$P_7$

Table 7: Values of  $G[V_\lambda]$  and  $G[V \setminus V_\lambda]$  for  $T_3(2, 4, 6)$

$a$	$b$	$a$	$b$	$c$
0	arbitrary	0	0	$\equiv 1, 2 \pmod{3}$
1	$\equiv 0 \pmod{2}$		1	$\equiv 0 \pmod{2}$
2	$\not\equiv 0 \pmod{3}, 3, 6, 9, 12, 18, 21, 24, 36$		2	2, 3, 4, 6, 7, 8, 10, 11, 14, 16, 18, 19
3	$\equiv 0 \pmod{2}$		3	4, 8, 10
4	$4 \leq b \leq 19$ except for $b = 15$ ,		4	4, 5, 6, 8, 10, 11, 12, 14, 16
	$\equiv 2, 4 \pmod{6}$ with $20 \leq b \leq 46$	1	5	6
5	$\equiv 2, 4 \pmod{6}$ with $8 \leq b \leq 32, 6, 18$		2	2, 4, 6, 8, 10, 14, 16, 18
6	6, 7, 8, 10, 11, 12, 14, 16	2	3	4

(a) Values  $a, b$  ( $b \geq a$ ), such that  $\text{Sun}(a, b)$  is R-AVD (b) Values  $a, b, c$  ( $c \geq b \geq a$ ), such that  $\text{Sun}(a, b, c)$  is R-AVD

Table 8: Values for R-AVD Suns

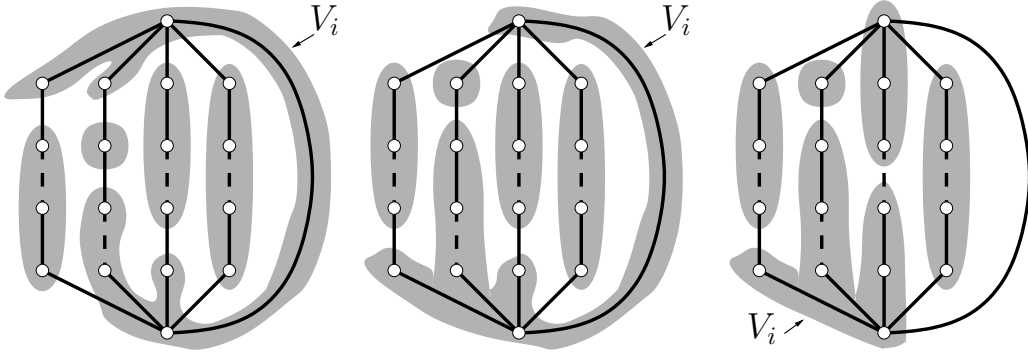


Figure 5: Removing the empty branch

**Proof.** Consider a balloon  $B(b_1, \dots, b_k, 0)$  and a decomposition  $\lambda = (\lambda_1, \dots, \lambda_p)$  of  $n = b_1 + \dots + b_k + 2$  (cf. Figure 5). The branch of width 0 means that it exists an edge between the two roots. The existence of such a branch is essential (i.e., guarantees connectedness of a part  $G[V_i]$ ) only if the part  $G[V_i]$  is a tree containing both roots of the balloon. Thus, the other parts  $V_j$ ,  $j \neq i$ , are all situated on the branches. We may then move them in such a way that for one root,  $V_i$  contains exactly one of its neighbours, that means the other root. And then, we may move one of the  $V_j$ ,  $j \neq i$ , on this root. Now, the edge of the branch with width zero is no more necessary and may be removed.  $\square$

**Theorem 19** *An R-AVD  $k$ -balloon has maximum degree at most 5. This bound is tight.*

The proof of Theorem 19 is based on Lemma 20 to 22.

**Lemma 20** *The two 5-balloons  $B(3, 2, 2, 1, 1)$  and  $B(4, 2, 1, 1, 1)$  are R-AVD.*

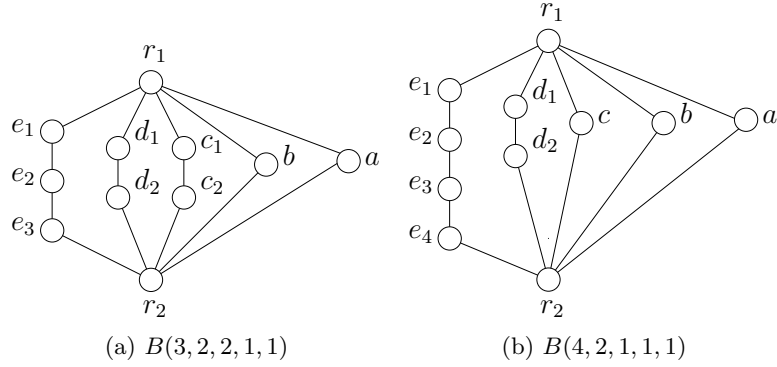


Figure 6: RAVD 5-balloons

**Proof.** The proof is based on Remark 6. The values of  $V_\lambda$  are given in the Table 9a for  $B(3, 2, 2, 1, 1)$  and Table 9b for  $B(4, 2, 1, 1, 1)$ , using vertex labelings from Figure 6. Please remark that

- $B(2, 2, 1, 1)$  contains  $\text{Cat}(3, 5)$  as a spanning tree and then is R-AVD,
- $B(4, 2, 1, 1)$  contains  $\text{Cat}(3, 7)$  as a spanning tree and then is R-AVD,
- $B(4, 1, 1, 1)$  contains  $\text{Cat}(2, 7)$  as a spanning tree and then is R-AVD,
- $B(2, 1, 1, 1)$  contains  $\text{Cat}(2, 5)$  as a spanning tree and then is R-AVD.

$\square$

**Lemma 21** *A  $k$ -balloon with  $k \geq 7$ , cannot be R-AVD.*

**Proof.** We have proved in subsection 3.1 that an R-AVD tree is a path or a 3-pode.

Consider a  $k$ -balloon  $B(b_1, \dots, b_k)$  of order  $n = b_1 + \dots + b_k + 2$  with  $k \geq 7$ ,  $b_1 \geq \dots \geq b_k$ , and the integer  $\lambda = b_1 + 1$ . The only way to obtain two R-AVD subgraphs with sizes  $\lambda$  and  $n - \lambda$  is to separate the vertices into two trees. Then, at least one of these trees is a  $k'$ -pode with  $k' \geq 4$ .

Thus, a  $k$ -balloon with  $k \geq 7$  cannot be R-AVD.  $\square$

**Lemma 22** *A 6-balloon cannot be R-AVD.*

$\lambda$	$V_\lambda$	$G[V_\lambda]$	$G[V \setminus V_\lambda]$	$\lambda$	$V_\lambda$	$G[V_\lambda]$	$G[V \setminus V_\lambda]$
1	$\{a\}$	$P_1$	Cat (3, 7)	1	$\{a\}$	$P_1$	$B(4, 2, 1, 1)$
2	$\{c_1, c_2\}$	$P_2$	Cat (2, 7)	2	$\{d_1, d_2\}$	$P_2$	$B(4, 1, 1, 1)$
3	$\{e_1, e_2, e_3\}$	$P_3$	$B(2, 2, 1, 1)$	3	$\{a, b, r_2\}$	$P_3$	Cat (3, 5)
4	$\{a, c_1, c_2, r_2\}$	$P_4$	Cat (3, 4)	4	$\{e_1, e_2, e_3, e_4\}$	$P_4$	$B(2, 1, 1, 1)$
5	$\{a, b, c_1, c_2, r_2\}$	Cat (2, 3)	$P_6$	5	$\{a, b, d_1, d_2, r_2\}$	Cat (2, 3)	$P_6$

(a) Values of  $G[V_\lambda]$  and  $G[V \setminus V_\lambda]$  for  $B(3, 2, 2, 1, 1)$  (b) Values of  $G[V_\lambda]$  and  $G[V \setminus V_\lambda]$  for  $B(4, 2, 1, 1, 1)$

Table 9: Values of  $G[V_\lambda]$  and  $G[V \setminus V_\lambda]$  for  $B(3, 2, 2, 1, 1)$  and  $B(4, 2, 1, 1, 1)$

**Proof.** We suppose there exists an R-AVD 6-balloon  $B_{min} = B(b_1, \dots, b_6)$  of order  $n = b_1 + \dots + b_6 + 2$  as small as possible.

Let  $\lambda$  be an integer, with  $\lambda \leq \frac{n}{2}$ . We want to split  $B_{min}$  into two R-AVD subgraphs  $G[V_\lambda]$  and  $G[V \setminus V_\lambda]$  of respective orders  $\lambda$  and  $n - \lambda$ .

Suppose that  $G[V_\lambda]$  is a path consisting of inner vertices of a branch of width  $b_i > \lambda$ . Then the 6-balloon  $B(b_1, \dots, b_{i-1}, b_i - \lambda, b_{i+1}, \dots, b_6)$  must be R-AVD. Because this 6-balloon has a smaller order than  $B_{min}$ , it is impossible.

The two remaining possibilities is to have a branch of width  $b_i = \lambda$  or three branches of respective widths  $b_{i_1}, b_{i_2}, b_{i_3}$ , with  $b_{i_1} + b_{i_2} + b_{i_3} + 1 = \lambda$ , and  $T_3(b_{i_1}, b_{i_2}, b_{i_3})$  R-AVD.

Now, consider the values from 1 to 6 for  $\lambda$ :

For  $\lambda \in \{1, 2, 3, 4, 6\}$ , the only possibility is to have a branch of width  $\lambda$ .

For  $\lambda = 5$ , the two possibilities are  $G[V_\lambda] = \text{Cat}(2, 3)$  or there is a branch with width 5.

Then, the two candidates for  $B_{min}$  are  $B1 = B(1, 1, 2, 3, 4, 6)$  and  $B2 = B(1, 2, 3, 4, 5, 6)$ .

- $B1 = B(1, 1, 2, 3, 4, 6)$   
Consider  $\lambda = 5$ . The only possibility for  $G[V_\lambda]$  is Cat (2, 3), using the two branches of width 1 and those of width 2. Then the remaining graph is  $T_3(3, 4, 6)$  which is not R-AVD. Thus,  $B1$  is not R-AVD.
- $B2 = B(1, 2, 3, 4, 5, 6)$   
Consider  $\lambda = 7$ . The only possibility for  $G[V_\lambda]$  is Cat (3, 4), using the branches of width 1, 2 and 3. The remaining graph is  $T_3(4, 5, 6)$  which is not R-AVD. Thus,  $B2$  is not R-AVD.

□

### 3.3.2 Other results

**Lemma 23** Let  $b_1, \dots, b_k$  be positive integers with  $b_1 \geq \dots \geq b_k \geq 1$ . If  $B(b_1, \dots, b_k)$  is AVD, then

$$\sum_{j=i+1}^k b_j \leq 2b_i \text{ for } i = 1, \dots, k-1.$$

**Proof.** Consider the decomposition  $\lambda = (b_i + 1, \dots, b_i + 1, r) = (\lambda_1, \dots, \lambda_l)$  of  $n$  and its  $B(b_1, \dots, b_k)$ -realization  $(V_1, \dots, V_l)$ . Note that each part  $G[V_j]$  of order  $b_i + 1$  contains at least one root of the balloon. Let  $s$  denote the sum of terms of the decomposition  $\lambda$  corresponding to the minimal set of parts that cover inner vertices of branches of widths  $b_i, \dots, b_k$ . Since  $i \leq k-1$ , at least one summand in  $s$  is equal to  $b_i + 1$ , and so at least one root is covered. On the other hand, there are two roots, hence at most two summands in  $s$  are equal to  $b_i + 1$ . Besides that, the summand  $r$  can be involved in  $s$ .

If  $s$  has just one summand equal to  $b_i + 1$ , then  $\sum_{j=i}^k b_j + 1 \leq (b_i + 1) + r \leq 2(b_i + 1)$  and  $\sum_{j=i+1}^k b_j \leq b_i < 2b_i$ .

If, however,  $s$  has two summands equal to  $b_i + 1$ , then  $\sum_{j=i}^k b_j + 2 \leq 2(b_i + 1) + r < 3(b_i + 1)$  and  $\sum_{j=i+1}^k b_j \leq 2b_i$ .  $\square$

In the next lemma, we denote by  $BP(b_1, b_2, \dots, b_k)$  the graph formed by attaching a path of length  $b_1$  to a  $(k - 1)$ -balloon  $B(b_2, \dots, b_k)$ .  $BP(b_1, b_2, \dots, b_k)$  may also be view as a spanning subgraph of the  $k$ -balloon  $B(b_1, b_2, \dots, b_k)$  after removing an edge joining the branch of width  $b_1$  to a root.

**Lemma 24**  $BP(b_1, 2, 1, 1)$  with  $b_1 \geq 0$  and  $BP(b_1, 1, 1, 1)$  with  $b_1 \equiv 0 \pmod{2}$  are  $R$ -AVD.

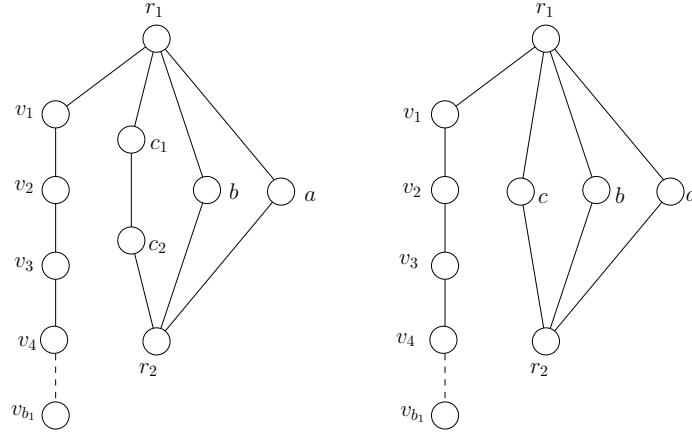


Figure 7:  $BP(b_1, 2, 1, 1)$  and  $BP(b_1, 1, 1, 1)$

**Proof.** The proof is based on Lemma 6. The values of  $V_\lambda$  for  $BP(b_1, 2, 1, 1)$  are given in Table 10, and for  $BP(b_1, 1, 1, 1)$  in Table 11. These tables use the vertex labeling from Figure 7.

Please remark that

- for  $BP(b_1, 2, 1, 1)$ ,  $\lambda \geq 6 \Rightarrow b_1 \geq \lambda$ ,
- for  $BP(b_1, 1, 1, 1)$ ,  $\lambda \geq 5 \Rightarrow b_1 \geq \lambda$ ,
- $BP(b_1, 2, 1)$  and  $BP(b_1, 1, 1)$  are traceable,
- $BP(b_1, 1, 1)$  is traceable,
- $\lambda$  odd  $\Rightarrow \lambda - 5$  even
- $\lambda$  and  $b_1$  even  $\Rightarrow b_1 - \lambda$  even.

$\square$

## 4 Strongly Recursively AVD graphs

In this section, we give a characterization of SR-AVD graphs. This characterization, based on the exclusion of two induced subgraphs, leads a polynomial algorithm to check if a graph is SR-AVD or not.

$\lambda$	$V_\lambda$	$G[V_\lambda]$	$G[V \setminus V_\lambda]$
1	$\{a\}$	$P_1$	$BP(b_1, 2, 1)$
2	$\{c_1, c_2\}$	$P_2$	$BP(b_1, 1, 1)$
3	$\{a, b, r_2\}$	$P_3$	$P_{b_1+3}$
4	$\{a, c_1, c_2, r_2\}$	$P_4$	$P_{b_1+2}$
5	$\{a, b, c_1, c_2, r_2\}$	$\text{Cat}(2, 3)$	$P_{b_1+1}$
6	$\{a, b, c_1, c_2, r_1, r_2\}$	$B(2, 1, 1)$	$P_{b_1}$
$\geq 7$	$\{v_{b_1}, v_{b_1-1}, \dots, v_{b_1-\lambda+1}\}$	$P_\lambda$	$BP(b_1 - \lambda, 2, 1, 1)$

Table 10: Values of  $G[V_\lambda]$  and  $G[V \setminus V_\lambda]$  for  $BP(b_1, 2, 1, 1)$

$\lambda$		$V_\lambda$	$G[V_\lambda]$	$G[V \setminus V_\lambda]$
1		$\{a\}$	$P_1$	$BP(b_1, 1, 1)$
2		$\{a, r_2\}$	$P_2$	$\text{Cat}(2, b_1 + 1)$
3		$\{a, b, r_2\}$	$P_3$	$P_{b_1+2}$
4	$b_1 = 0$	$\{a, b, r_1, r_2\}$	$C_4$	$P_1$
	$b_1 = 2$	$\{v_2, v_1, r_1, c\}$	$P_4$	$P_3$
	$b_1 \geq 4$	$\{v_{b_1}, v_{b_1-1}, v_{b_1-2}, v_{b_1-3}\}$	$P_4$	$BP(b_1 - 4, 1, 1, 1)$
5		$\{a, b, c, r_1, r_2\}$	$B(1, 1, 1)$	$P_{b_1}$
$\geq 5$ and odd		$\{a, b, c, r_1, r_2, v_1, \dots, v_{\lambda-5}\}$	$BP(\lambda - 5, 1, 1, 1)$	$P_{b_1-\lambda+5}$
$\geq 5$ and even		$\{v_{b_1}, \dots, v_{b_1-\lambda+1}\}$	$P_\lambda$	$BP(b_1 - \lambda, 1, 1, 1)$

Table 11: Values of  $G[V_\lambda]$  and  $G[V \setminus V_\lambda]$  for  $BP(b_1, 1, 1, 1)$ ,  $b_1 \equiv 0 \pmod{2}$

**Definition 25** A claw is a star isomorphic to  $K_{1,3}$ . A net is a graph obtained from a triangle by attaching to each vertex a new dangling edge (cf. Figure 8). A graph is said claw- and net-free if it has no induced subgraph isomorphic to either a claw or a net.

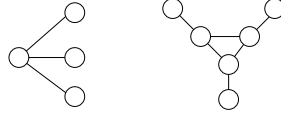


Figure 8: A claw and a net

We use the following theorem on connected claw- and net-free graphs :

**Theorem 26** [5] Any connected claw- and net-free graph is traceable.

Now, we may give a characterization of the SR-AVD graphs :

**Theorem 27** A connected graph  $G$  is SR-AVD if and only if it is claw- and net-free.

**Proof.** First of all, we may remark that both claw and net are not SR-AVD (and even not AVD). The sequence  $(2, 2)$  is not realizable in the claw and the sequence  $(3, 3)$  is not realizable in the net. Thus, if a graph is SR-AVD, it cannot contain a claw or a net as induced subgraph.

The proof of the sufficient condition is based on induction. Let  $P(n)$  be the following proposition: "Every connected claw- and net-free graph of order  $n$  is SR-AVD".  $P(1)$  is trivially true.

Suppose that  $P(i)$  is true for any  $1 \leq i \leq n-1$  and let  $G$  be a connected claw- and net-free graph of order  $n$ . Thus  $G$  is traceable (and hence AVD) for any decomposition  $\lambda = (\lambda_1, \dots, \lambda_k)$  of length  $k \geq 2$  admissible for  $G$  we can find (using a hamiltonian path in  $G$ ) a partition  $(V_1, \dots, V_k)$  of the vertex set of  $G$  in which  $G[V_i]$  is connected and of order  $\lambda_i$  for  $i = 1, \dots, k$ . Since clearly each  $G[V_i]$  is claw-free, net-free and of order smaller than  $n$ , by the induction hypothesis it is SR-AVD. Thus,  $P(n)$  is true for any  $n \geq 1$ .  $\square$

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